1/30/26 Collect HW

Last time main result:

Theorem

If $T: \mathbb{R}^m \to \mathbb{R}^n$ satisfies the linearity property (*) then it is a linear transformation defined by the $n \times m$ matrix

$$A = \left[T(\vec{e}_1) \;\middle|\; T(\vec{e}_2) \;\middle|\; \cdots \;\middle|\; T(\vec{e}_m) \right]$$

Another Question (which we'll come back to later):
if $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, does there exist a lin. transformation $T^{-1}: \mathbb{R}^n \to \mathbb{R}^m$ that "undoes" T (ie inverse)?
So if $T(\vec{x}) = \vec{y}$, we always have $T^{-1}(\vec{y}) = \vec{x}$?
If so, and if A defines T, what matrix defines T^{-1}? We'll call it A^{-1}.

We'll just start by looking at the case $m = n = 2$, but we'll come back to the general question.
See discussion in beginning of §2.1 (see p. 43)

Ex $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 &= y_1 \\ 3x_1 + 4x_2 &= y_2 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} x_1 + 2x_2 &= y_1 \\ 0x_1 - 2x_2 &= -3y_1 + y_2 \end{aligned}$$

$$\rightsquigarrow \quad \begin{aligned} x_1 + 2x_2 &= y_1 \\ 0x_1 + x_2 &= \tfrac{3}{2}y_1 - \tfrac{1}{2}y_2 \end{aligned}$$

$$\rightsquigarrow \quad \begin{aligned} x_1 + 0x_2 &= -2y_1 + y_2 \\ 0x_1 + x_2 &= \tfrac{3}{2}y_1 - \tfrac{1}{2}y_2 \end{aligned}$$

$$\begin{aligned} x_1 &= -2y_1 + y_2 \\ x_2 &= \tfrac{3}{2}y_1 - \tfrac{1}{2}y_2 \end{aligned}$$

So T^{-1} is defined by

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

(This is defined for <u>every</u> y_1, y_2)

We say A is <u>invertible</u>, and its <u>inverse</u> is $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$.

e.g. $A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$

and $A^{-1}\begin{bmatrix} 4 \\ 10 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 4 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Foreshadowing:

$$AA^{-1} = \underset{(2\times 2)}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}\underset{(2\times 2)}{\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}} = 2\times 2$$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and you can check $A^{-1}A = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

we'll do matrix mult. very soon.

But you can't always do this:

<u>Ex</u> $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

So $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{matrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{matrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\Rightarrow \quad \begin{aligned} x_1 + 2x_2 &= y_1 \\ 2x_1 + 4x_2 &= y_2 \end{aligned}$$

Solve this system

$x_1 + 2x_2 = y_1$

$0 + 0 = -2y_1 + y_2$

(only defined for special y_1, y_2, namely satisfying $-2y_1 + y_2 = 0$)

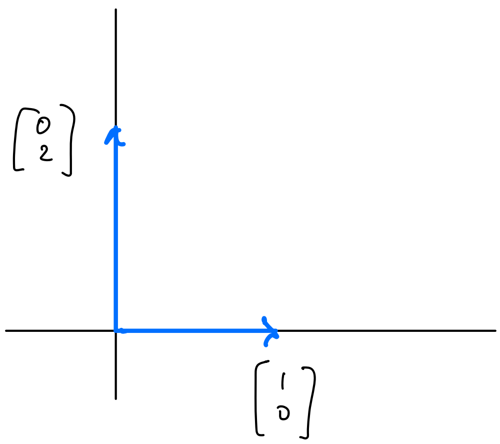
e.g. $A\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ and $A\begin{bmatrix}-4\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

so there can't be an A^{-1} sending $\begin{bmatrix}0\\0\end{bmatrix}$ to both $\begin{bmatrix}-2\\1\end{bmatrix}$ and $\begin{bmatrix}-4\\2\end{bmatrix}$!

(since T^{-1} is supposed to be a <u>function</u>)

<u>Ex</u> (from book)

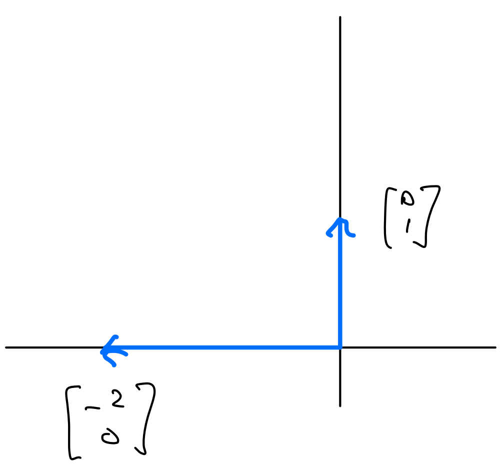
Consider the vectors $\begin{bmatrix}0\\2\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$



what does the lin transf T given by the matrix $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ do to these two vectors?

$$T\left(\begin{bmatrix}0\\2\end{bmatrix}\right) = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}-2\\0\end{bmatrix}$$

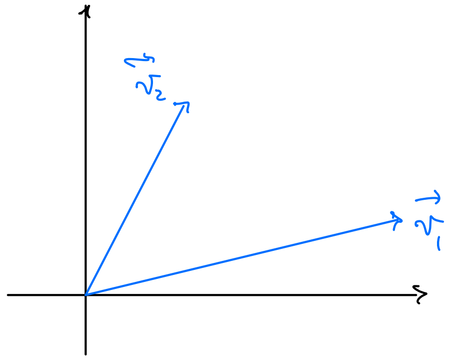
$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$



result:

So it rotates the picture by 90° counterclockwise!

<u>Ex</u> Let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ be as in this picture:



Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a lin transf with the property that

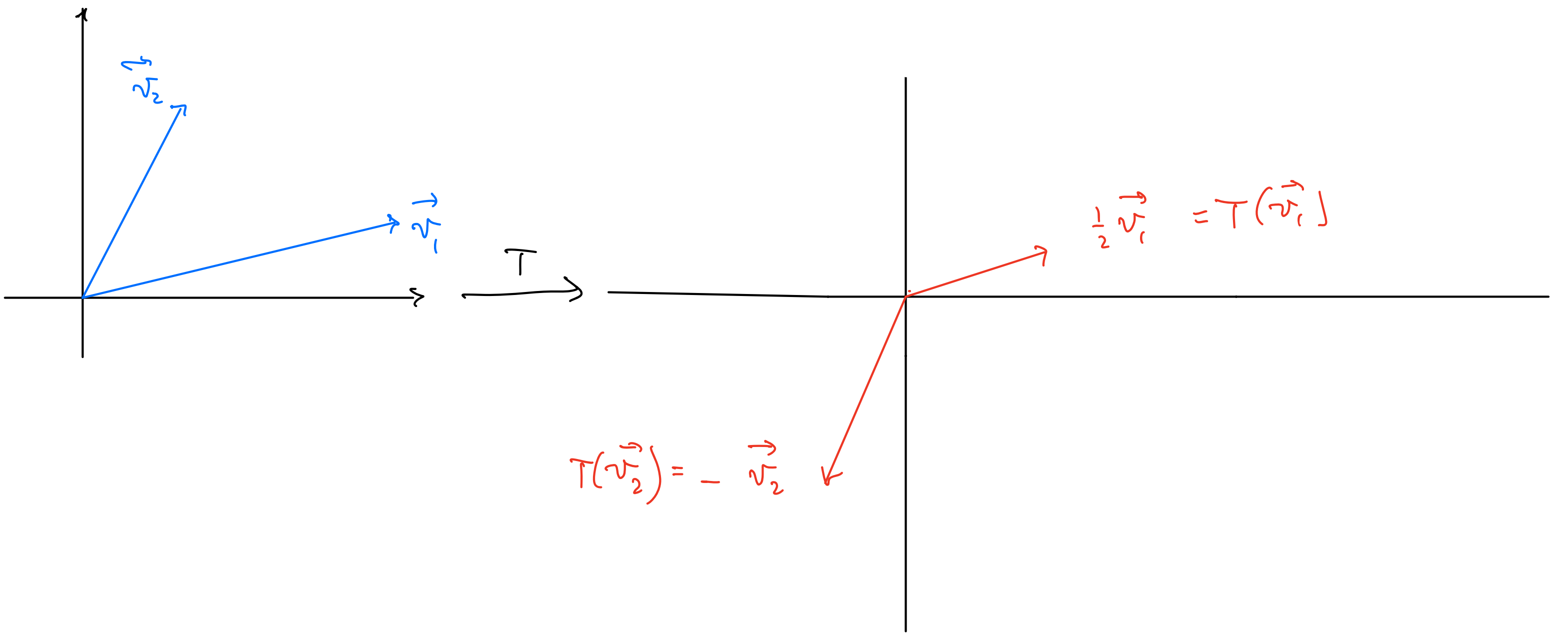
$$T(\vec{v}_1) = \tfrac{1}{2}\vec{v}_1$$
$$T(\vec{v}_2) = -\vec{v}_2$$

Sketch $T(\vec{v}_1)$, $T(\vec{v}_2)$ and $T(3\vec{v}_1 + 2\vec{v}_2)$.

First some algebra

$$T(3\vec{v}_1 + 2\vec{v}_2) = 3T(\vec{v}_1) + 2T(\vec{v}_2)$$

$$= \frac{3}{2}\vec{v}_1 - 2\vec{v}_2$$



$$T(3\vec{v}_1 + 2\vec{v}_2) = 3T(\vec{v}_1) + 2T(\vec{v}_2)$$